Centre for Theoretical Physics
Nijenborgh 4
9747 AG Groningen

TENTAMEN GENERAL RELATIVITY

friday, 26-08-2005, room 5118-156, 14.00-17.00

Indicate at the first page clearly your name, address, date of birth, year of arrival and at every other page your name.

Question 1

Consider a manifold with coordinates X^a $(a = 1, \cdots n)$. Let $X^a(u)$ be a curve in that manifold with evolution paramter u.

(1.1) How do you define the absolute derivative $\frac{\mathrm{D}}{\mathrm{Du}} V_a$ of a covariant vector field $V_a(x)$?

(1.2) The absolute derivative of the metric g_{ab} along any curve is zero:

$$\frac{\mathrm{D}}{\mathrm{Du}} g_{ab} = 0\,. \tag{1}$$

Use this fact to derive an expression for the Christoffel symbol Γ^c_{ab} in terms of the metric and the derivative of the metric.

Consider the action for a massive particle:

$$S = -\frac{1}{2}\int_{u_1}^{u_2} du \{\frac{1}{e}\dot{X}^a \dot{X}^b g_{ab} + m^2 e\}\,. \tag{2}$$

Here $e > 0$ is the Einbein and m a mass parameter.

(1.3) Show that the action (2) is invariant under the worldline reparametrizations $u' = u'(u)$.

(1.4) Use the Euler-Lagrange equations

$$\frac{\partial L}{\partial X^a} - \frac{d}{du}\frac{\partial L}{\partial \dot{X}^a} = 0 \tag{3}$$

with the Lagrangian L given by (2) to derive the geodesic equation for $X^a(u)$. Hint: use the gauge $e = 1/m$.

Question 2

Consider the Robertson-Walker metric for $k = 1$ (we take $c = 1$)

$$ds^2 = dt^2 - R(t)^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\}\,. \qquad (4)$$

For the energy-momentum tensor of a perfect fluid the Einstein equations lead to the following relations between the function $R(t)$, the mass density $\rho(t)$ and the pressure $p(t)$:

$$\frac{(\dot{R})^2 + 1}{R^2} = \tfrac{1}{3}\kappa\rho\,, \qquad (5)$$

$$\dot{\rho} + 3(p + \rho)\frac{\dot{R}}{R} = 0\,. \qquad (6)$$

The dot indicates a differentiation with respect to t and $\kappa = 8\pi G$ (G is Newton's constant).

We first consider the situation of a Friedmann universe with non-relativistic matter, i.e. $p = 0$.

(2.1) Show that ρR^3 is constant.

(2.2) Show that $R(t)$ satisfies the differential equation

$$\left(\frac{dR}{dt}\right)^2 + 1 = \frac{A^2}{R}\,, \qquad (7)$$

where A is a constant. We impose the boundary condition that $R = 0$ at $t = 0$. Show that, given these boundary conditions, the solution of the differential equation (7) is given by the equations

$$R = \tfrac{1}{2}A^2(1 - \cos\psi)\,, \qquad (8)$$

$$t = \tfrac{1}{2}A^2(\psi - \sin\psi)\,, \qquad (9)$$

where ψ is a parameter. Give the graph of the function $R(t)$. Is this universe open or closed?

We next consider the situation of a Friedmann universe with ultra-relativistic matter, i.e. $p = \frac{1}{3}\rho$.

(2.3) Show that ρR^4 is constant.

(2.4) Determine R as a function of t. Take as boundary condition that $R = 0$ at $t = 0$. Give the graph of the function $R(t)$. Is this universe open or closed? Hint: follow the same steps as in question (2.2).

Question 3

Consider the Schwarzschild metric (we take $c = 1$)

$$ds^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{10}$$

(3.1) In the Newtonian limit the time-time component of the metric g_{00} is related to Newton's gravitational potential ϕ as follows ($c = 1$):

$$g_{00} = 1 + 2\phi\,. \tag{11}$$

Use this equation to show that the parameter m in the Schwarzschild metric (10) is given by

$$m = GM\,, \tag{12}$$

where G is Newton's constant and M is the mass of the central object.

(3.2) Calculate the expressions for the Christoffel symbols $\Gamma^r_{tt}, \Gamma^r_{rr}, \Gamma^r_{\theta\theta}$ and $\Gamma^r_{\phi\phi}$.

For constant r and $\theta = \pi/2$ the metric (10) leads to the following geodesic equations

$$(1 - \frac{2m}{r})\dot{t} = k\,, \tag{13}$$

$$r^2\dot{\phi} = h\,, \tag{14}$$

$$\frac{m}{r^2}(\dot{t})^2 - r(\dot{\phi})^2 = 0\,, \tag{15}$$

with k and h constant. The dot $\cdot$ indicates differentation with respect to the parameter u of the geodesic.

(3.3) A light-ray follows a geodesic around a Schwarzschild black hole for constant $r = r_0$ and $\theta = \pi/2$. Determine the value of r_0.

(3.4) Determine the coordinate time Δt that the light ray needs to complete one circular orbit.